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Quantum Logic for Observation of Physical Quantities

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[**Motivation**]

DQL (dynamic quantum logic) is one of the fields of QL, and it can express many important notions of quantum physics.
But there is a limit to what can be expressed in DQL.
Therefore, in this study, we will add the concepts of **measurement of physical quantities** that is difficult to be expressed in DQL, and show some theorems.

[**Contents**]

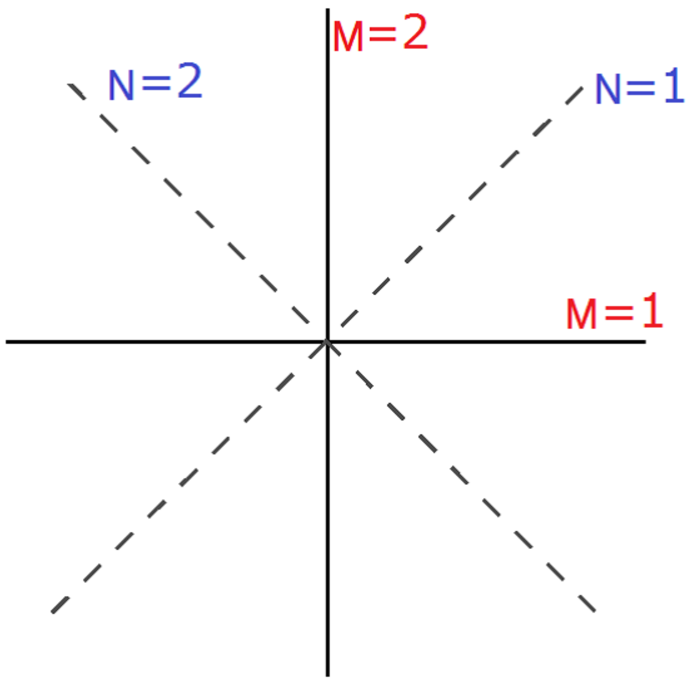
1 Backgrounds

Quantum logic (QL) is the field which deal with propositions about physical values of particle or about states in quantum mechanics.

Ex: "In the current state, velocity of the particle is 20".

A **Hilbert space** is used for state space of a particle in quantum mechanics.

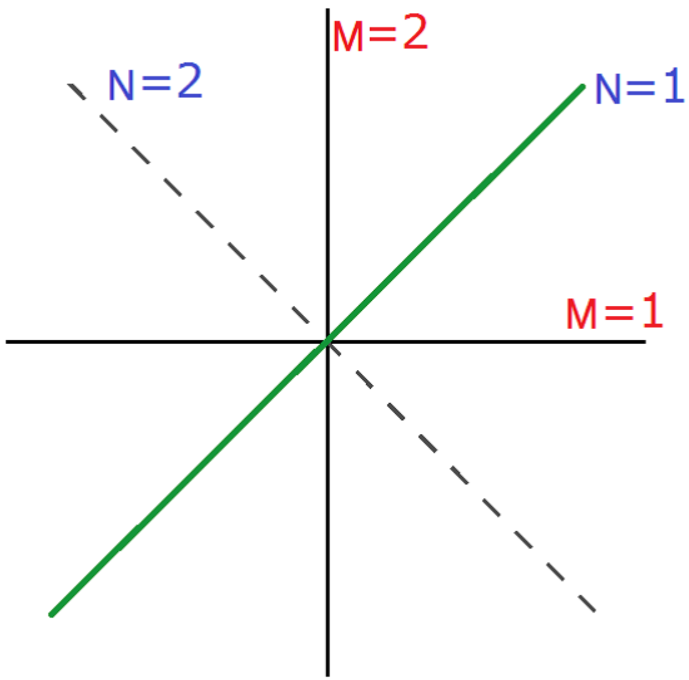
Example: 2D Hilbert space



· The state of the particle is represented by **a one-dimensional subspace** passing through the origin.

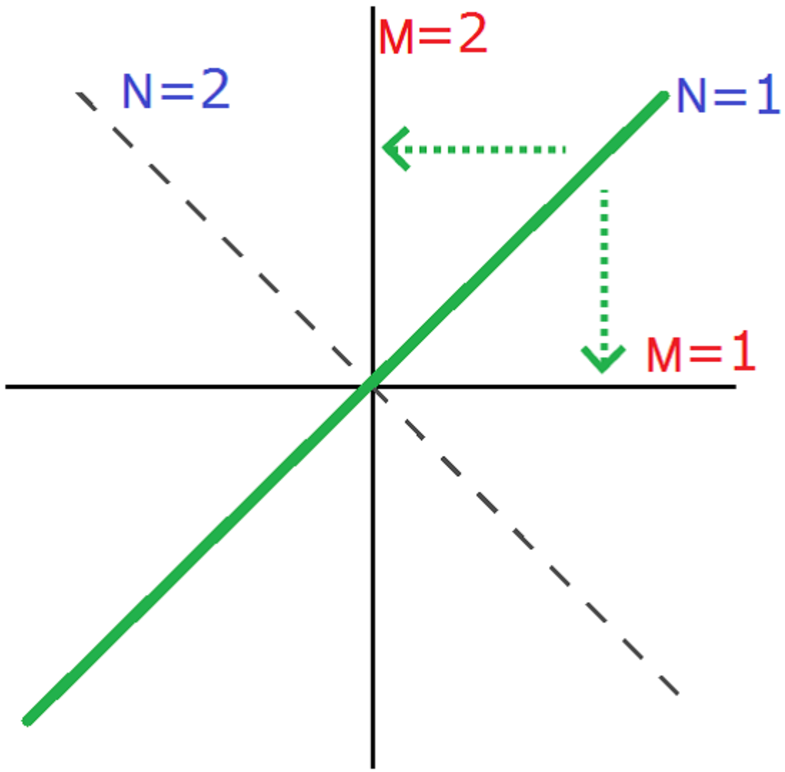
· Each physical quantity is associated with one basis respectively, and specific values of it are associated with each base.

Example: 2D Hilbert space

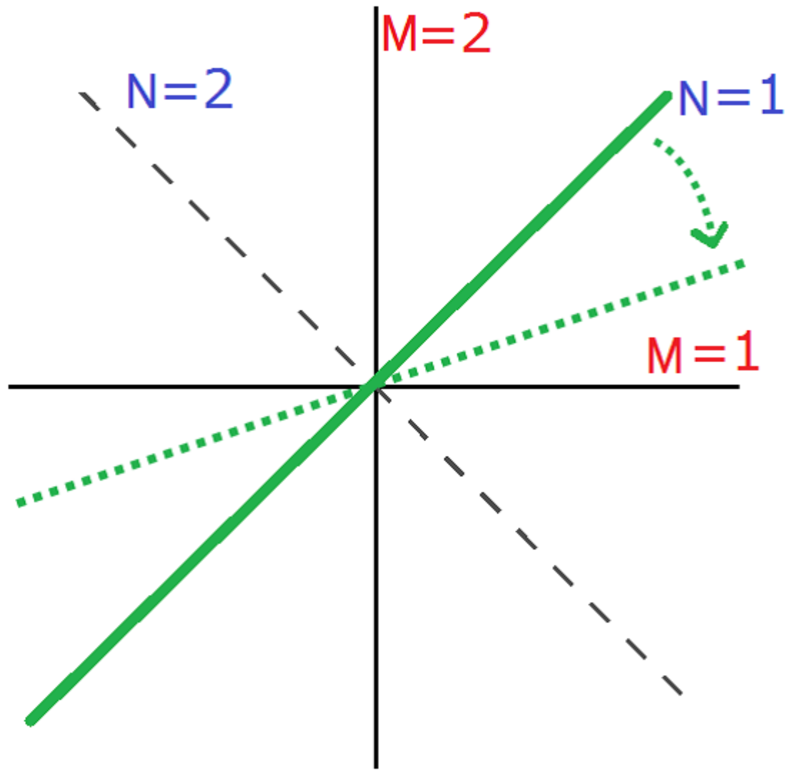


· The probability of obtaining a specific value depends on the component of the state.

· If physical quantity M is measured in green state, we get 1 with 50% and 2 with 50%. If N is measured in green state, we get 1 with 100%.



Projections by measurement



Unitary evolutions

· Basically, there are two kinds of transition of particle state in quantum physics.

2 Dynamic quantum logic

Quantum logic (Orthomodular logic)

- Orthogonality

Dynamic quantum logic (A. Baltag and S. Smets. 2004 -)

- Orthogonality
- Projection
- Unitary transformation

Language of dynamic quantum logic

A denote formulas, and π denote actions
(same as dynamic logic).

$$A ::= p \mid \bot \mid \neg A \mid A \wedge B \mid [\pi]A \mid \Box A$$
$$\pi ::= U \mid \pi^{\dagger} \mid \pi \cup \pi \mid \pi ; \pi \mid A?$$

U : Atomic symbols for unitary transformation. $\{U, V, ...\}$

$[\pi]A$: A is always true after executing action π

Symbol $\dagger$ denote *conjugate* of a transformation.

Dynamic quantum frame $\langle X, \{\xrightarrow{Y?}\}_{Y \subseteq X}, \{\xrightarrow{U}\}_{U \in \mathcal{U}}\rangle$

- X is a non-empty set.
- $\{\xrightarrow{Y?}\}_{Y \subseteq X}$ is a set of binary relations $Y?(Y \subseteq X)$ on X
- $\{\xrightarrow{U}\}_{U \in \mathcal{U}}$ is a set of binary relations $U'(U' \in \{U, V, ...\})$ on X

Intuitively, X is a set of quantum states, $\{\xrightarrow{Y?}\}_{Y \subseteq X}$ and $\{\xrightarrow{U}\}_{U \in \mathcal{U}}$ represent projections and unitary transformations.

Binary relation $\not\perp$ on X. (Non-orthogonality)

$x \not\perp y \stackrel{\text{def}}{=}$ there exists $Y \subseteq X$ such that $x(Y?)y$ or $y(Y?)x$.

We write $x \perp y$ if not $x \not\perp y$.

Conditions for dynamic quantum frame

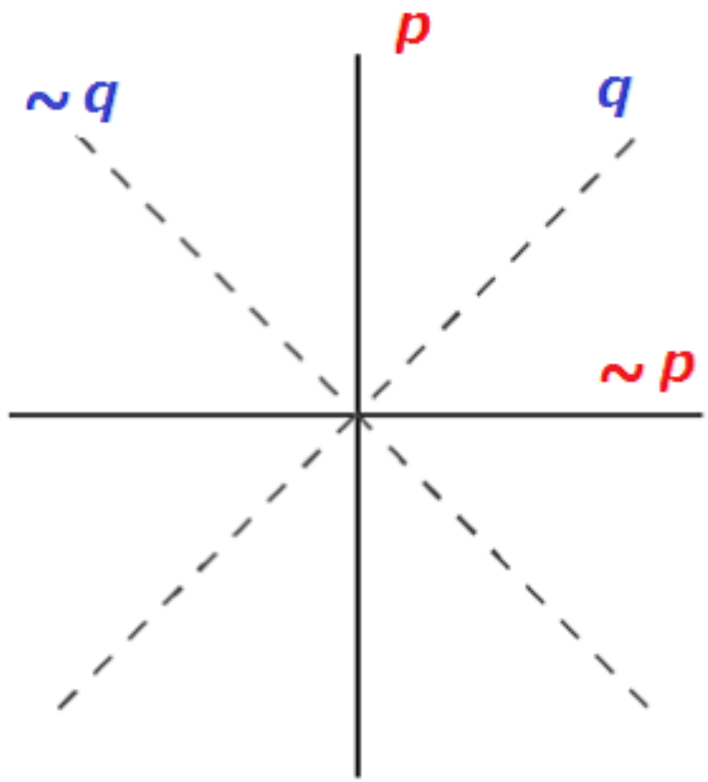
1. There is no $x, y \in X$ such that $x(\phi?)y$. For all $x \in X$, $x(X?)x$.
2. For all $x, y, z \in X$, if $x(Y?)y$ and $x(Y?)z$, then $y = z$. (Partial functionality of $P?$)
3. If $x \in Y$, then $x(Y?)x$. (Adequacy)
4. For all $x, y \in X$, if $Y \subseteq X$ is testable and $x(Y?)y$, then $y \in Y$. (Repeatability)
5. For all $Y, Z \subseteq X$, if Y and Z are testable and $Y?; Z? = Z?; Y?$, then $Y?; Z? = (Y \cap Z)?$. (Compatibility)
6. Let (R) be $(Y?), (U)$ or $(U^\dagger)$. If $x(R)y$ and $y \not\perp z$, then there exists $w \in X$ such that $z(R^\dagger)w$ and $w \not\perp x$. (Self-adjointness)
7. For all $x \in X$ and U, $\exists! y \in X$ such that $x(U)y$. (Functionality for U)
8. For all $x \in X$ and $U^\dagger$, $\exists! y \in X$ such that $x(U^\dagger)y$. (Functionality for $U^\dagger$)
9. For all $x, y \in X$, $x(U)y$ iff $y(U^\dagger)x$. (Bijectivity)
10. For all $x, y \in X$, there exists $z \in X$ such that $x \not\perp z$ and $z \not\perp y$. (Universal accessibility)

Dynamic quantum model $\langle X, \rightarrow_{P?}, \rightarrow_{U}, V \rangle$

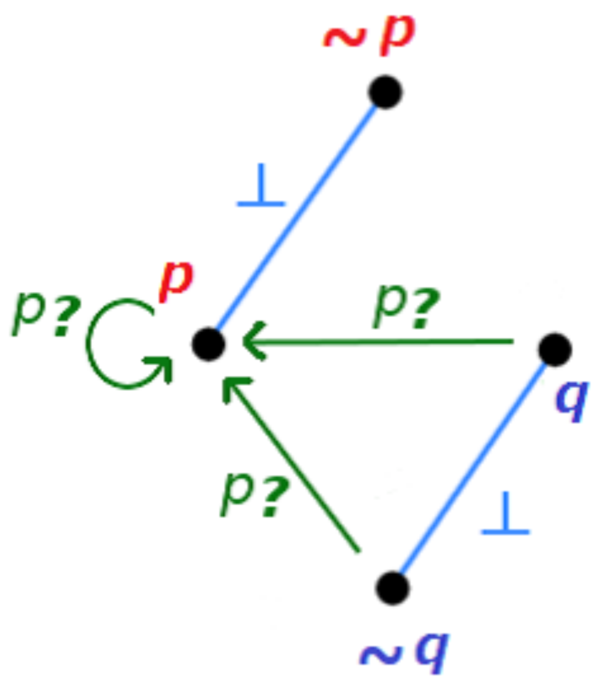
V is a function assigning each propositional variable p to a subset of X. The truth of formulas ($x \models A \Leftrightarrow x \in \|A\|$) are defined as follows:

$\|p\| = V(p)$
$\|\bot\| = \phi$
$\|A \wedge B\| = \|A\| \cap \|B\|$
$\|\neg A\| = \|A\|^c$
$\|\Box A\| = \{x \in X|$ for all $y \in X$, if $x \not\perp y$, then $y \in \|A\|\}$
$\|[A?]B\| = \{x \in X|$ for all $y \in X$, if $x(Y?)y$, and $Y = \|A\|$, then $y \in \|B\|\}$
$\|[U]A\| = \{x \in X|$ for all $y \in X$, if $x(U)y$, then $y \in \|A\|\}$
$\|[\pi_1; \pi_2]A\| = \|[\pi_1][\pi_2]A\|$
$\|[\pi_1 \cup \pi_2]A\| = \|[\pi_1]A\| \cap \|[\pi_2]A\|$
$\|[B?^\dagger]A\| = \|[B?]A\|$
$\|[U^\dagger]A\| = \{x \in X|$ for all $y \in X$, if $x(U^\dagger)y$, then $y \in \|A\|\}$
$\|[(\pi_1; \pi_2)^\dagger]A\| = \|[\pi_2^\dagger; \pi_1^\dagger]A\|$
$\|[(\pi_1 \cup \pi_2)^\dagger]A\| = \|[\pi_1^\dagger \cup \pi_2^\dagger]A\|$
$\|[\pi^{\dagger\dagger}]\|A = \|[\pi]A\|$

Example

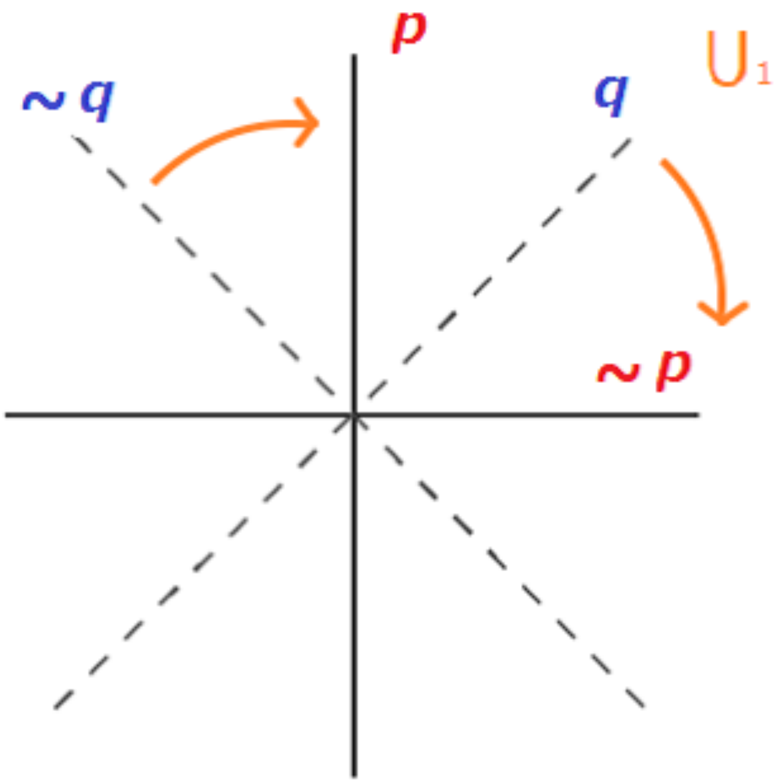


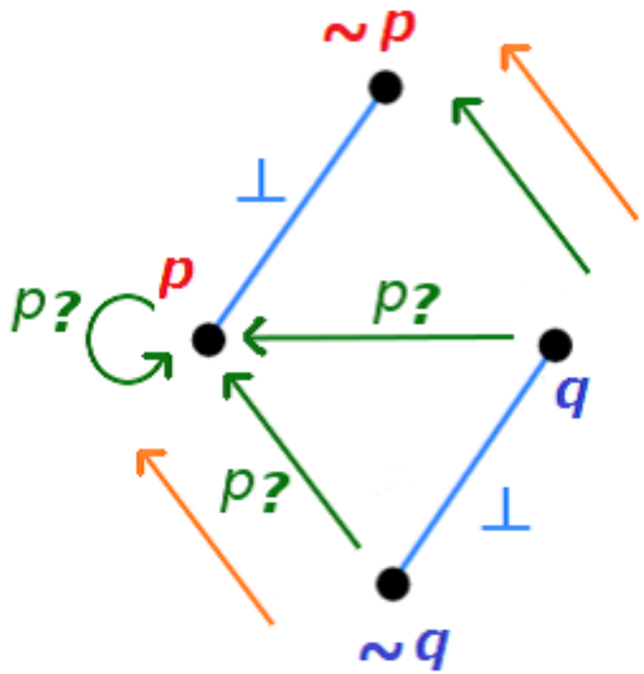
2D Hilbert space



One of the corresponding models
(only some relations are written)

Example





2D Hilbert space

One of the corresponding models
(only some relations are written)

PDQL (propositional dynamic quantum logic). Baltag, A., Smets, S.(2004-)

All the axioms and rules of classical dynamic logic
(Necessitation Rule): If A is provable, then infer $[\pi]A$
(Kripke Axiom): $[\pi](A \to B) \to ([\pi]A \to [\pi]B)$
(Test Generalization): If $A \to [C?]B$ is provable for all C, then infer $A \to \Box B$
(Testability Axiom): $\Box A \to [B?]A$
(Partial Functionality): $\neg[A?]B \to [A?]\neg B$
(Adequacy): $A \wedge B \to \langle A?\rangle B$
(Repeatability): $[A?]A$ for all testable formulas A
(Universal Accessibility): $\langle \pi \rangle \Box\Box A \to [\pi']A$
(Unitary Functionality): $\neg[U]A \leftrightarrow [U]\neg A$
(Unitary Bijectiviity 1): $A \leftrightarrow [U; U^\dagger]A$
(Unitary Bijectiviity 2): $A \leftrightarrow [U^\dagger; U]A$
(Adjointness): $A \to [\pi]\Box\langle \pi^\dagger \rangle \Diamond A$
(Substitution Rule): If A is provable, then infer $A[p/B]$
(Compatibility Rule): For all testable formulas A, B and every propositional variable p which does not appears in A, B, if $\langle A?; B?\rangle p \to \langle B?; A?\rangle p$ is provable, then infer $\langle A?; B?\rangle p \to \langle (A \wedge B)?\rangle p$

Expressing closed subspaces

For $Y \subseteq X$,
$Y^{\perp} \stackrel{\text{def}}{=} \{x \in X | \text{for all } y \text{ in } Y, x \perp y\}$.

A set $Y \subseteq X$ is is called **testable** set if $Y^{\perp\perp} = Y$.

$$\sim A \stackrel{\text{def}}{=} \Box\neg A$$
$$T(A) \stackrel{\text{def}}{=} \Box\Box(\sim\sim A \to A) \qquad (\|A\| \text{ is testable})$$

$Y \sqcup Z \stackrel{\text{def}}{=} (Y^{\perp} \cap Z^{\perp})^{\perp}$
(Quantum disjunction)
expresses spanned space of Y and Z.

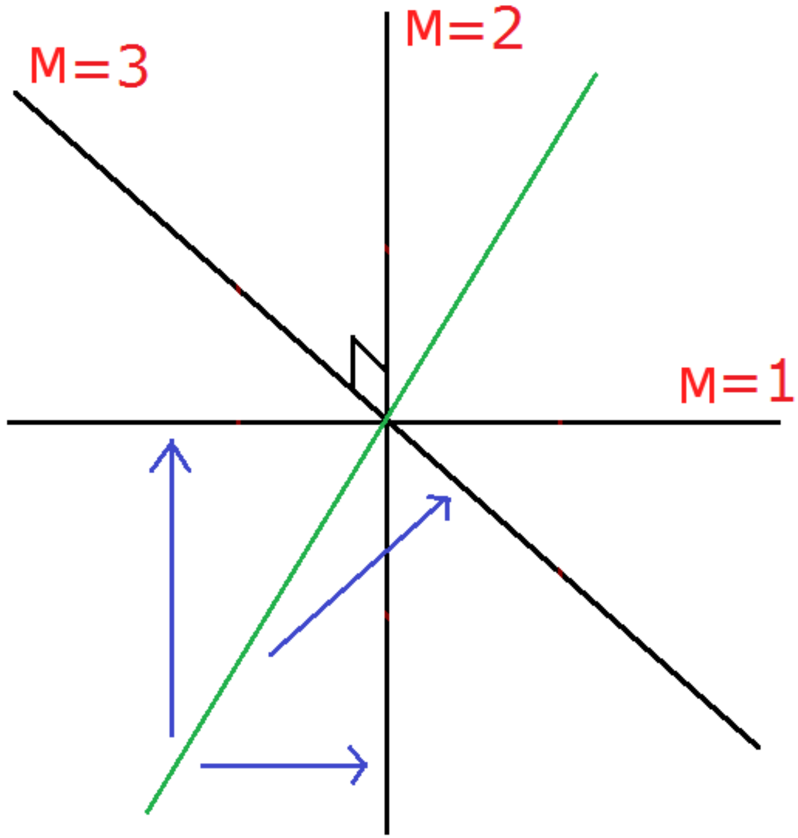
3 Modality for measurement

Let M be a physical quantity whose eigenvalues are not degenerate.

Propositions like
"After a measurement of M, (whatever the result), A is true"
are difficult to represented by dynamic quantum logic.

This proposition may be represented by
$[(M = 1)? \cup (M = 2)? \cup (M = 3)? \cup ...]A$

Example: 3D Hilbert space



Modality $[(M=1)?\cup(M=2)?\cup(M=3)]$

Problems

1. In a dynamic quantum frame, there is no definition of an orthonormal basis.

2. If an orthonormal basis has infinite elements, the set of formulas that represent bases $\{B_1, B_2, B_3, ...\}$ will be an infinite set. However, infinite chain $B_1? \cup B_2? \cup ...$ is not allowed.

Problems

1. In a dynamic quantum frame, there is no definition of an orthonormal basis.

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↓ Therefore, we add the definition for orthonormal basis and some modal symbols.

A set $Ob \subseteq \mathcal{P}(X)$ is *orthonormal basis* of $\langle X, \rightarrow_{P?}, \rightarrow_U \rangle$ if Ob satisfies following conditions.

1. If $S \in Ob$, then S is testable.
   (Testability of 1D-subspace)

2. If $S \in Ob$, and for all testable subset $Y \subseteq X$, if $S \cap Y \neq \phi$, then $S \subseteq Y$.
   (Atomicity of bases)

3. If $S \in Ob$, $T \in Ob$ and $S \neq T$, then for all $x \in S$ and $y \in T$, $x \perp y$.
   (Orthogonality)

4. $\bigsqcup_{Y \in Ob} Y = X$.
   (Completeness of basis)

The set of **propositional variables for orthonormal basis** is defined as follows.

$Bp = \{s, t, ...\} \subset \{p, q, ...\}$ where both Bp and $\{p, q, ...\} - Bp$ are infinite sets.

New modal operator $\boxdot$ is introduced and is regard as a quantification of $[s?](s \in Bp)$.

Intuitively, $\boxdot A$ is corresponds to $[s? \cup t? \cup ...]A$.

$$\| \boxdot A \| \stackrel{\mathrm{def}}{=} \{x \in X \mid \text{for all } y \in X \text{ and for all } s \in Bp, \text{ if } x(s?)y, \text{ then } y \in \|A\|\}$$

We say $\langle X, \to_{P?}, \to_U, Ob\rangle$ is a **EDQ-frame** if it satisfies following conditions.

1. $\langle X, \to_{P?}, \to_U\rangle$ is a dynamic quantum frame.
2. Ob is an orthonormal basis of $\langle X, \to_{P?}, \to_U\rangle$.

We say $\langle X, \to_{P?}, \to_U, Ob, V\rangle$ is a **EDQ-model** if it satisfies following conditions.

1. $\langle X, \to_{P?}, \to_U, Ob\rangle$ is a EDQ-frame.
2. V is a function assigning each propositional variable p (including $s \in Bp$) to a subset of X which satisfies $V(s) \in Ob$.
3. For every $Y \in Ob$, there exists $s \in Bp$ and $V(s) = Y$.

New logic **PDQLB** (PDQL with basis) is defined by adding the following rules and axioms to **PDQL**.

Rules

If A is provable and $p \notin Bp$, then infer $A[p/B]$

(Substitution Rule for **PDQLB**)

If $A \to [s?]B$ is provable for all s, then infer $A \to \boxdot B$.

(Test Generalization for Bp)

Axioms

$\boxdot A \to [s?]A$ (Testability Axiom for Bp)

$T(s)$ (Testability of Basis)

$s \wedge A \wedge T(A) \to \Box\Box(s \to A)$ (Atomicity of Basis)

$s \to t \vee \sim t$ (Orthogonality of Basis)

$\neg \boxdot \bot$ (Completeness of Orthonormal Basis)

Some important formulas for basis can be proved in **PDQLB**.

$s \wedge A \to \boxdot A$ (Eigenstate)

$\boxdot A \to \boxdot \boxdot A$ (Repeatability of measurement)

A proof of $\boxdot A \to \boxdot \boxdot A$.

1. $s \wedge A \to \boxdot A$ (s does not appears in A)
2. From necessitation rule, Kripke axiom and 1, $[s?]s \wedge [s?]A \to [s?] \boxdot A$
3. From repeatability, testability of s and 2, $[s?]A \to [s?] \boxdot A$
4. $\boxdot A \to [s?]A$
5. From 3 and 4, $\boxdot A \to [s?] \boxdot A$
6. As s does not appears in A, from test generalization for Bp and 5, $\boxdot A \to \boxdot \boxdot A$

Theorem 3.1 All axioms and rules of **PDQLB** is valid in all EDQ-models.

Theorem 3.2 (Complete axiomatization for orthonormal bases) If all axioms and rules of **PDQLB** is valid in a T-complete dynamic quantum model $\langle X, \to_{P?}, \to_U, V \rangle$, then $\{\|s\| | s \in Bp\}$ is an orthonormal basis of $\langle X, \to_{P?}, \to_U \rangle$.

T-complete: For all testable sets Y, there exists A such that $Y = \|A\|$.

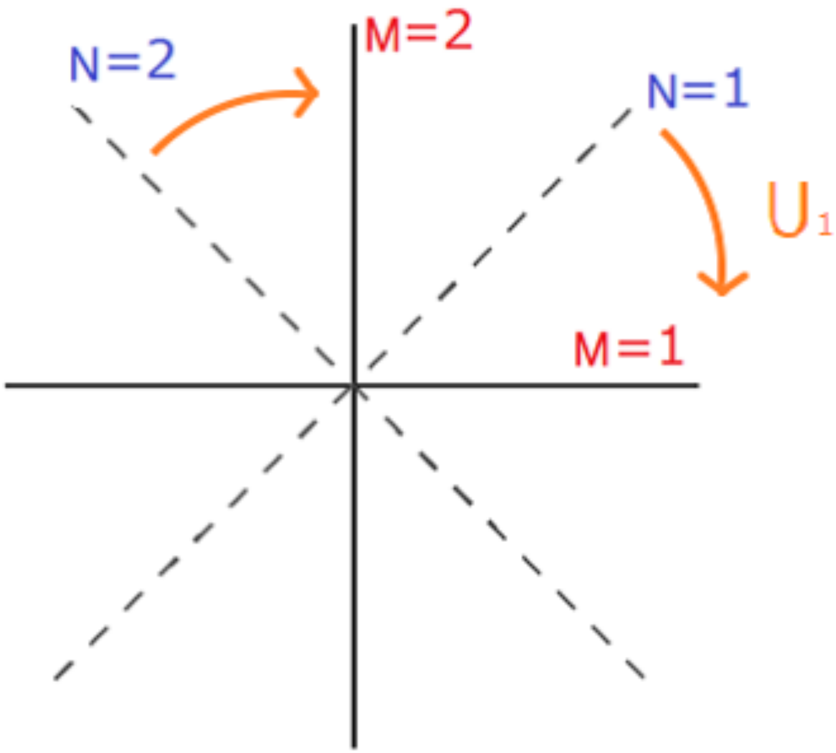
Multiple physical quantities

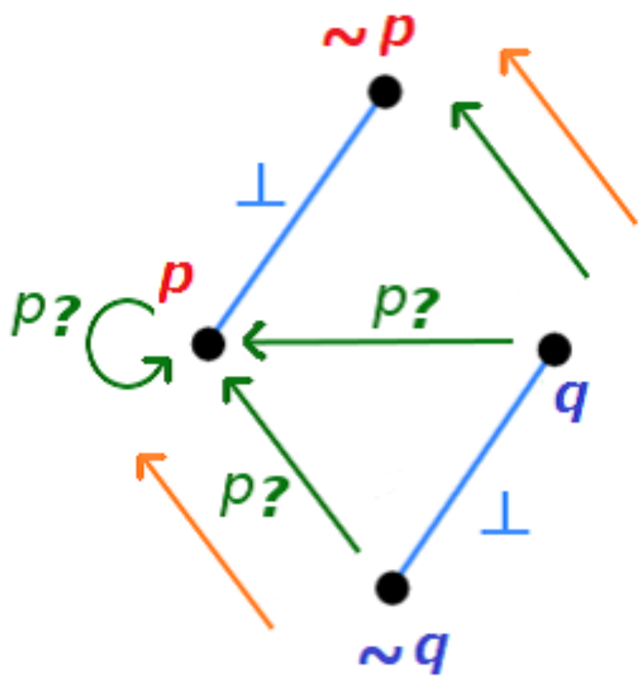
Each (non-degenerate) physical quantity corresponds to an orthonormal basis Ob. In a Hilbert space, another orthonormal basis can be constructed by unitary transformations from Ob.

Definitions of x_U, Y_U and Ob_U.

If $x(U)y$, then $y = x_U$
$Y_U = \{x \in X | \exists y \in Y \text{ and } y(U)x\}$
$Ob_U = \{Y_U \subseteq X | Y \in Ob\}$

Example: 2D Hilbert space





Theorem 3.3 If Ob is an orthonormal basis of dynamic quantum frame $\langle X, \to_{P?}, \to_U \rangle$, then Ob_U is also an orthonormal basis of $\langle X, \to_{P?}, \to_U \rangle$.

$\| \boxdot_U A \| \stackrel{\text{def}}{=} \{x \in X|$ for all $s \in Bp$, if $x(([U^\dagger]s)?)y$, then $y \in \|A\|\}$

New logic **PDQLBU** (**PDQLB** with unitary transformations) is defined by adding the following rules to **PDQLB**.

(Test Generalization for $U^\dagger(Bp)$): If $A \to [([U^\dagger]s)?]B$ is provable for all s, then infer $A \to \boxdot_U B$

(Testability Axiom for $U^\dagger(Bp)$): $\boxdot_U A \to [([U^\dagger]s)?]A$

(Testability of Basis): $T([U^\dagger]s)$

(Atomicity of Basis):
$[U^\dagger]s \wedge A \wedge T(A) \to \Box\Box([U^\dagger]s \to A)$

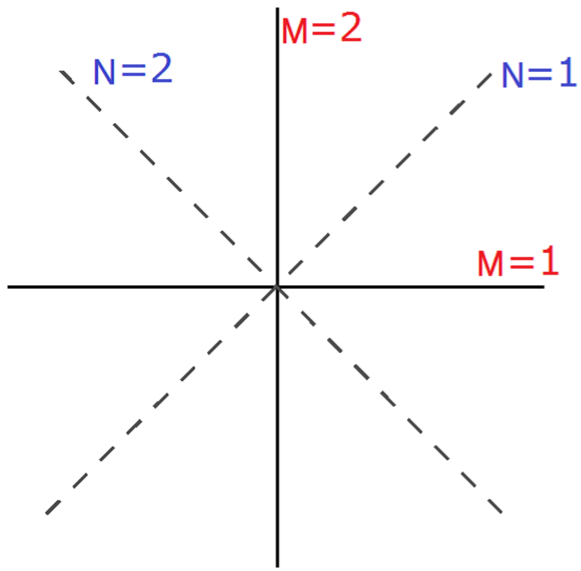
(Orthogonality of Basis): $[U^\dagger]s \to [U^\dagger]t \vee \sim [U^\dagger]t$

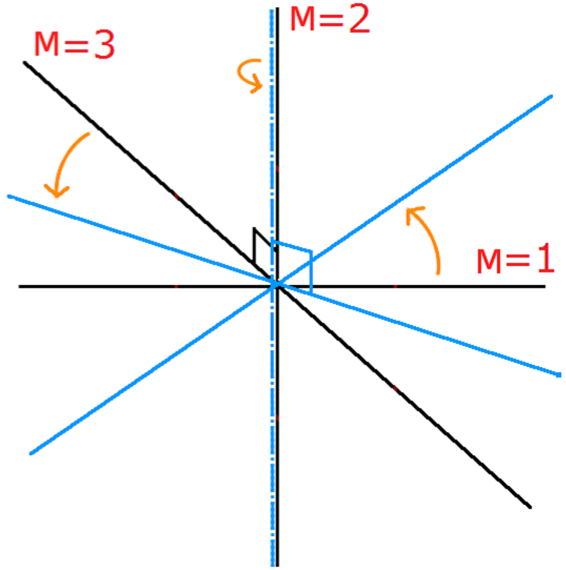
(Completeness of Orthonormal Basis): $\neg \boxdot_U \perp$

Theorem 3.4 All axiom and rules of **PDQLBU** is valid in all EDQ-model.

Theorem 3.5 (Complete axiomatization for orthonormal basis) If all axioms of **PDQLBU** is valid in T-complete dynamic quantum model $\langle X, \rightarrow_{P?}, \rightarrow_U, V\rangle$, then $\{\|[U^\dagger]s\| | s \in Bp\}$ is an orthonormal basis of $\langle X, \rightarrow_{P?}, \rightarrow_U\rangle$.

Mutually unbiased bases





Mutually unbiased bases   Not mutually unbiased

Two orthonormal bases are **mutually unbiased bases** if each basis equally contains all the component of the other orthogonal basis. (An orthonormal basis for position and an orthonormal basis for momentum are mutually unbiased bases in an infinite dimensional Hilbert space).

However, degree of non-orthogonality cannot be expressed by the framework of this study.

Orthonormal bases Ob and Ob' are defined as **quasi-mutually unbiased bases** of dynamic quantum frame $\langle X, \rightarrow_{P?}, \rightarrow_U \rangle$ if Ob and Ob' satisfies following conditions.

1. For all $y \in Y \in Ob'$ and $Z \in Ob$, there exists $z \in Z$ such that $y \not\perp z$.
2. For all $y \in Y \in Ob$ and $Z \in Ob'$, there exists $z \in Z$ such that $y \not\perp z$.

Axioms for quasi-mutually unbiased bases

$$T(A) \wedge \neg \boxdot_U \neg \boxdot A \rightarrow \Box\Box A$$
$$T(A) \wedge \neg \boxdot \neg \boxdot_U A \rightarrow \Box\Box A$$

Theorem 3.6 In a EDQ-model $\langle X, \rightarrow_{P?}, \rightarrow_U, Ob, V\rangle$, if Ob and Ob_U are quasi-mutually unbiased bases of $\langle X, \rightarrow_{P?}, \rightarrow_U\rangle$, then the axioms for quasi-mutually unbiased bases of U are valid in $\langle X, \rightarrow_{P?}, \rightarrow_U, Ob, V\rangle$.

Theorem 3.7 In a T-complete dynamic quantum model $\langle X, \rightarrow_{P?}, \rightarrow_U, V\rangle$, if $\{\|s\| | s \in Bp\}$ and $\{\|[U^\dagger]s\| | s \in Bp\}$ are orthonormal bases, and if axioms for quasi-mutually unbiased bases of U are valid, then $\{\|s\| | s \in Bp\}$ and $\{\|[U^\dagger]s\| | s \in Bp\}$ are quasi-mutually unbiased bases of $\langle X, \rightarrow_{P?}, \rightarrow_U\rangle$.

Future works

- Degeneracy of physical quantities

$$M_1 = \{A_1, A_2, \ldots\}$$
$$M_2 = \{B_1, B_2, \ldots\}$$

The conditions for M_j are almost the same as Ob but A_i and B_i do not necessarily satisfy **atomicity**.

Simultaneous observability $\quad [M_1][M_2]C \leftrightarrow [M_2][M_1]C \leftrightarrow \boxdot C$

- Degree of orthogonality

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Thank you for listening !